Regulated Fractional Variance Ratio Unit Root Tests

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Abstract

This article addresses unit root testing on regulated series through the variance ratio (VR) statistic of Nielsen (2009). The asymptotic distribution of the regulated VR statistic is developed with and without OLS detrending. Results of Cavaliere and Xu (2011) are extended by also developing the asymptotic distribution of regulated series with a linear trend. Asymptotic local power is analyzed for various choices of the fractional integration parameter d. It is shown that power performance depends crucially on the length of the regulating interval. When the interval is sufficiently wide the results in Nielsen (2009) are replicated.

Keywords: Regulated time series, Fractionally integrated time series, Fractional Brownian motion, Variance Ratio Statistic, Unit root testing, Hypothesis testing

1. Introduction

Regulated (bounded) integrated time series ought to be of significant practical importance to time series econometricians. Indeed! Series such as nominal interest rates, target zone exchange rates and unemployment rates are just some of many series which are regulated either below, above, or both. Nevertheless, these series are often treated as their unregulated counterparts. The presence of bounds is simply disregarded in theoretical and empirical works. The danger however is that this disregard for the nature of data renders traditional integration methods theoretically unjustifiable in the presence of regulated integrated processes.

Until recently, Cavaliere (2005), Granger (2010), and Cavaliere and Xu (2011) were the only serious efforts to develop a theory for regulated integrated processes. Particularly important is the seminal work of Cavaliere (2005) in which he develops asymptotic distributions for well known unit root test statistics when the driving series is a regulated I(1) process. These results were further developed and expounded on in Cavaliere and Xu (2011) by broadening the framework to allow for more general innovation structures.

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Nevertheless, these developments are still only concerned with bounded I(1) and do not address more general setups such as integrated processes of the form I(d) for some $d \in \mathbb{R}$. In this light Trokić (2013) extends the regulated I(1) framework and develops the limiting distribution for regulated I(d) processes under general innovation structures. He shows that unlike in the case of the regulated I(1) process where the limiting distribution is a regulated Brownian motion, the regulated I(d) process tends in distribution to a regulated fractionally integrated Brownian motion.

The paper by Trokić (2013) does not offer any discussion on how the regulated I(d) series may be useful in applied work. This paper proposes to fill this gap by developing limiting distributions for tuning parameter free variance ratio statistics for unit root tests as considered in Nielsen (2009). These non-parametric statistics have desirable power properties in addition to not suffering from ambiguous short-run dynamics estimators. Moreover, in order to carry over the results of Nielsen (2009) to the framework of bounded series, this paper also develops the first theoretical justification for the limiting distribution of an integrated bounded series with a linear time trend. Additionally, limiting distributions are also obtained in the case of OLS detrended series.

This paper is organized as follows. The next section will develop a model of a regulated fractionally integrated processes with a linear trend and fixed bounds. The assumptions required for the theoretical results to hold are outlined there as well. Section 3 states and discusses the main theoretical results. Section 4 presents simulation studies and Section 5 concludes.

2. Regulated Fractionally Integrated Processes

Regulated fractionally integrated processes generalizes the notion of regulated I(1) processes to regulated I(d) processes for some $d > -1/2$. Here, by a *regulated* process one means a series with bounds (possibly infinite) above, below, or both. That is, for some bounded interval $[\underline{b}, \overline{b}]$ with fixed bounds $\underline{b} < \overline{b}$, a series x_t is said to be regulated if $x_t \in [\underline{b}, \overline{b}]$ almost surely for all t. In the case of $d = 1$, Cavaliere and Xu (2011) refer to these series as bounded I(1), or BI(1) series. When d is not an integer however, these processes will be referred to as regulated fractionally integrated processes of order d, or RFI(d) in short.

A relatively general description of regulated fractionally integrated processes may be obtained as follows:

$$y_t = \phi y_{t-1} + \triangle_+^{-d} u_t, \quad \phi = 1 \tag{1}$$

$$u_t = v_t + \underline{\xi}_{d,t} - \overline{\xi}_{d,t} \tag{2}$$

$$v_t = \Psi(L)\epsilon_t, \quad \Psi(z) \equiv \sum_{j=0}^{\infty} \psi_j z^j \tag{3}$$

$$\tilde{x}_t^{(i)} = \delta_t^{(i)} \gamma^{(i)} + y_t, \quad i = 0, 1, 2 \tag{4}$$

Above, L is the usual lag operator, ϵ_t is a martingale difference sequence (MDS), and $\underline{\xi}_{d,t} \equiv \triangle_+^d \underline{\xi}_t$, and $\overline{\xi}_{d,t} \equiv \triangle_+^d \overline{\xi}_t$ where $\underline{\xi}_t$ and $\overline{\xi}_t$ are regulators which ensure that $\tilde{x}_t^{(i)} \in [\underline{b}, \overline{b}]$, see Harrison (1985). Both regulators are in fact non-negative and satisfy the following relations:

$$\underline{\xi}_t > 0 \quad \text{iff} \quad y_{t-1} + \triangle_+^{-d} v_t < \underline{b} - \delta_t^{(i)} \gamma^{(i)} \tag{5}$$

$$\overline{\xi}_t > 0 \quad \text{iff} \quad y_{t-1} + \triangle_+^{-d} v_t > \overline{b} - \delta_t^{(i)} \gamma^{(i)} \tag{6}$$

Notice the $\delta_t^{(i)} \gamma^{(i)}$ term in (4) - (6). The presence of this term bridges a gap in the current literature by addressing linear trends in regulated series. Although Cavaliere and Xu (2011) and Carrion-i Silvestre and Gadea (2010) present discussions and suggestions on linear trends, thus far no theoretical justification has been brought forth. This paper aims to do this here.

Thus far the literature on this subject has dealt with fixed bounds of the form $[\underline{b}, \overline{b}]$. The presence of a linear trend however requires a modification of this construction so as to allow the bounds themselves to trend along with the series. Such a modification is not exactly trivial. One must make a distinction between a bounded series with trending bounds of the form $\Delta x_t \in [\underline{b}_t, \overline{b}_t]$ and $[\underline{b}_t, \overline{b}_t] \neq [\underline{b}_s, \overline{b}_s]$ for $t \neq s$, and one where the series is trending but the bounds satisfy $\Delta x_t \in [\underline{b}, \overline{b}]$ for some fixed interval $[\underline{b}, \overline{b}]$.[1] This paper is concerned with the latter scenario which can be visualized by a trending series bounded by two parallel lines which trend along at the same rate as the series itself. In other words, the bounds of the series are characterized by a local linear trend. Local linear drifts were first examined by Nabeya and Sørensen (1994), Haldrup and Hylleberg (1995), and Haldrup (1996), and they rest on the idea of scaling the trending parameter by an appropriate controlled rate so as to account for sample size growth. The focus of said papers was on standard Brownian motions requiring the controlled rate to be $T^{-1/2}$. In this paper however, as will be seen later, the presence of fractional Brownian motions requires this controlled rate to be $T^{-(1/2-d)}$. As a first step in this direction, define $\delta_t^{(0)} = 0$ and $\gamma^{(0)} = 0$ when $i = 0$, $\delta_t^{(1)} = 1$ and $\gamma^{(1)} = \gamma_0$ when $i = 1$, and when $i = 2$, define $\delta_t^{(2)}$ as the 1×2 vector $[1, t]$ and $\gamma^{(2)}$ as the 2×1 vector $[\gamma_0, \gamma_1]^\top$. The trending bounds considered in this paper can now be defined as $[\underline{b}_t^{(i)}, \overline{b}_t^{(i)}]$ where $\underline{b}_t^{(i)} = \underline{b} + \delta_t^{(i)} \gamma^{(i)}$ and $\overline{b}_t^{(i)} = \overline{b} + \delta_t^{(i)} \gamma^{(i)}$, for some fixed bounds $\underline{b}$ and $\overline{b}$. In other words, when $i = 0, 1$, the above model reduces to that looked at by Trokić (2013) and all results considered there continue to hold here.

What remains to be addressed are the fractional sums. Recall that a general fractional process $\{z_t\}$ of order d is defined as

$$(1-L)^d z_t = u_t, \quad u_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}, \quad t = 1, 2, \ldots \tag{7}$$

[1] For a good exposition of other possible constructions, see Carrion-i Silvestre and Gadea (2010).

where $d > -1/2$, ϵ_t are zero-mean, finite variance, IID random variables, and $(1-L)^d$ is defined by the Maclaurin series:

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(-d+j)}{\Gamma(-d)\Gamma(j+1)} L^j$$

In empirical work however, one does not observe the values of a series for $t \leq 0$. For this reason, the theory which follows assumes that only terms with a positive time index are of interest. It can be shown (see Appendix 2 of Wang et al. (2002)) that when the time index is restricted to positive integers, z_t in (7) reduces to

$$z_t = (1-L)_+^{-d} u_t \equiv \triangle_+^{-d} u_t = \sum_{k=0}^{t-1} c_k^{(d)} u_{t-k}$$

where

$$c_0^{(0)} = 1, \quad c_k^{(0)} = 0, \quad k \geq 1, \quad c_k^{(d)} = \frac{\Gamma(d+k)}{\Gamma(d)\Gamma(k+1)}, \quad k \geq 0$$

Finally, let $W(t)$ denote a standard Wiener process and recall that a type II fractional Brownian motion $B_d(t)$ for $d > -1/2$ is defined as:

$$B_d(t) = \int_0^t (t-s)^{d-1} dW(s), \quad B_d(0) = 0, \quad 0 \leq t \leq 1$$

For a good reference on the differences between type I and type II fractional Brownian motions, see Davidson and Hashimzade (2009).

2.1. Model Assumptions

The theoretical results of this paper rest on a set of relatively weak assumptions. Some of these assumptions are necessary for the functional central limit theorem for fractional processes to hold, while others ensure that the bounds of the regulated series are properly defined so as to account for sample size. Throughout the remainder of the paper the following assumptions hold:

Assumptions.

(a) $\{\epsilon_t, \mathcal{F}_t\}$ *is a MDS with respect to some filtration* $\mathcal{F}_t$ *and* $E\{\epsilon_t^2|\mathcal{F}_t\} = \sigma^2 < \infty$.

(b) $\sup_{t\in\mathbf{Z}} E\{|\epsilon_t|^p\} < \infty$ *for* $p > \max\{2, 2/(2d+1)\}$ *and* $d > -1/2$.

(c) $\sum_{j=0}^{\infty} |\psi_j| < \infty$, $\sum_{j=0}^{\infty} j|\psi_j| < \infty$, *and* $b_\psi = \sum_{j=0}^{\infty} \psi_j \neq 0$.

(d) $\sup_{t\in\mathbf{Z}} E\{|\underline{\xi}_t|^p\} < \infty$ *and* $\sup_{t\in\mathbf{Z}} E\{|\overline{\xi}_t|^p\} < \infty$, *for* p *as in (b).*

(e) $\gamma_1 = \gamma_c T^r$ *where* γ_c *is a fixed constant and* $r = d - 1/2$.

(f) $\left(\kappa(d)T^{(d+1/2)}\right)^{-1} \underline{b}_t^{(i)} = \underline{c}_t^{(i)}(d) + o(1)$

$\left(\kappa(d)T^{(d+1/2)}\right)^{-1} \overline{b}_t^{(i)} = \overline{c}_t^{(i)}(d) + o(1)$

for $i = 0, 1, 2$, *where* $\kappa(d) = \frac{b_\psi \sigma}{\Gamma(d+1)}$ *and* $\underline{c}^{(i)}(d) \neq \overline{c}^{(i)}(d)$, $d > -1/2$.

(g) $\left(\kappa T^{1/2}\right)^{-1} \underline{b}_t^{(i)} = \underline{c}_t^{(i)} + o(1)$ *and* $\left(\kappa T^{1/2}\right)^{-1} \overline{b}_t^{(i)} = \overline{c}_t^{(i)} + o(1)$ *for* $i = 0, 1, 2$, *where* $\kappa = b_\psi \sigma$ *and* $\underline{c}^{(i)} \neq \overline{c}^{(i)}$.

A quick reflection on the assumptions is in order. Assumptions (a) through (c) establish regularity conditions for the error terms and are primarily required to invoke (fractional) functional central limit theorems. These conditions are very similar to those imposed by Cavaliere (2005) and Cavaliere and Xu (2011). Particularly important here is Assumption (b). Although this moment condition can be very strong when d is close to $-1/2$, it is in fact necessary as shown in Johansen and Nielsen (2012) and has a long standing tradition in the literature since Davydov (1970). Several important FCLTs for fractional processes such as Marinucci and Robinson (2000), Davidson and De Jong (2000), Tanaka (1999), Wang et al. (2003), and Lee and Shie (2004), rest on this moment condition. Assumption (c) is also quite important when defining u_t as a linear process of ϵ_t. In fact, an alternative specification of the second condition in (c) is also possible with $\sum_{j=0}^{\infty} j^{1/2-d}|\psi_j| < \infty$, see Phillips and Solo (1992) for a discussion when $d = 0$ and Wang et al. (2003) when $d > -1/2$. On the whole, the assumptions above allow for a relatively flexible dependence structure on u_t which includes for example stationary and invertible ARMA processes. Assumption (d) is the adaptation of assumption $\mathcal{B}_1$ in Cavaliere and Xu (2011) to fractionally integrated series. Assumption (e) is introduced in order to justify a linear trend in a regulated system. This is suggested by Cavaliere and Xu (2011) and is imposed here so as to avoid the situation where a series is completely absorbed by or diverges from the fixed bounds $[\underline{b}, \overline{b}]$. A theoretical justification is summarized in Lemma 1 although for now, it suffices to note that without (e), assumption (f) fails to hold when $i = 2$. In particular, assumptions (e) and (f) are the two conditions which drive a major contribution of this paper and represent the first theoretical attempts at introducing linear trends to regulated integrated series. It is helpful to notice that when $i = 0, 1$, the assumptions above, with the exception of (e) are in fact the same assumptions found in Trokić (2013). Formally then,

Definition 1. *A process* $\{\tilde{x}_t^{(i)}\}$ *satisfying equations* (1) - (6) *and assumptions (a) - (f) will be called a generalized regulated fractionally integrated process of degree d, or a generalized RFI(d) process in short.*

It is also important to discuss the relationship between $b_t^{(i)}$ and $c_t^{(i)}(d)$. Abstracting then from the bar notation for the moment, assumption (f) above says that fixing b and setting $i = 0$, c can be defined as $b = \left(\kappa(d)T^{(d+1/2)}\right) c(d)$. This implies that for $i = 0, 1, 2$ the following recursive relationship (similarly for the upper limit) holds as well.

$$
\begin{aligned}
\underline{b}_t^{(i)} &= \left(\kappa(d)T^{(d+1/2)}\right)\underline{c}_t^{(i)}(d)\\
&= \underline{b} + \delta_t^{(i)}\gamma^{(i)}\\
&= \left(\kappa(d)T^{(d+1/2)}\right)\underline{c}(d) + \delta_t^{(i)}\gamma^{(i)}
\end{aligned}
$$

Also, define $c_0^{(i)}(d) \in [\underline{c}_t^{(i)}, \overline{c}_t^{(i)}](d)$ as $\left(\kappa(d)T^{(d+1/2)}\right)^{-1}\gamma_0 = c_0^{(i)}(d)$. This setup ensures that all points including the initial point $\tilde{x}_0^{(i)}$ fall within the appropriate bounds.

Finally, assumption (g) was introduced in Cavaliere (2005) and Cavaliere and Xu (2011) as a way of defining distributional bounds for regulated time series. The following relations (which will be needed later) establish the link between the distributional bounds on standard regulated series and their fractional counterparts.

$$
c_t^{(i)}(d) = \Gamma(d+1)^{-1}T^{-d}c_t^{(i)}
$$

2.2. Asymptotic Distribution of the RFI(d) Process

Recall the definition of a regulated fractional Brownian motion.

Definition 2. *Let $z(t) = B_d(t)$ be a stochastic process on $\mathcal{C}$. Fix the bounds $\underline{b}$ and $\overline{b}$. If $z(0) \in [\underline{b}, \overline{b}]$, then there exist continuous functions $l(t)$ and $q(t)$ such that the process $B_d^{\underline{b},\overline{b}}(t) \equiv (z(t) + l(t) - q(t)) \in [\underline{b}, \overline{b}]$. The functions $l(t)$ and $q(t)$ are called regulators and the function $B_d^{\underline{b},\overline{b}}(t)$ will be called a "regulated type II fractional Brownian motion" with bounds at $[\underline{b}, \overline{b}]$.*

Let $\Rightarrow$ denote weak convergence and define the continuous time approximation of a RFI(d) process $\{\tilde{x}_t^{(i)}\}$ on the cadlag space $\mathcal{D}[0,1]$ as:

$$
\tilde{x}_T^{(i)}(t) = \left(\left(\frac{b_\psi \sigma}{\Gamma(d+1)}\right)^2 T^{2(d+1/2)}\right)^{-1/2}\left(\tilde{x}_{[Tt]}^{(i)} - \tilde{x}_0^{(i)}\right), \quad t \in [0,1] \tag{8}
$$

Trokić (2013) obtained the limiting distribution for $\tilde{x}_t^{(i)}$ for $i = 0, 1$. The aim here is to generalize the result above to also include the case when $i = 2$ and the regulated series exhibits a linear trend. This is the main result of this section and it can be formalized as follows:

Theorem 1. *Let $\tilde{x}_t^{(i)}$ be a RFI(d) process and fix the bounds $[\underline{b}, \overline{b}]$. If $\mathcal{D}[0,1]$ is endowed with the Skorohod topology, for all $i = 0, 1, 2$, if $\tilde{x}_0^{(i)} \in [\underline{b}_t^{(i)}, \overline{b}_t^{(i)}]$, then as $T \to \infty$, $\tilde{x}_T^{(i)}(t) \Rightarrow \chi_t^{(i)}(d) + B_{d+1}^{\underline{c}_t^{(i)}(d), \overline{c}_t^{(i)}(d)}(t)$ for any $d > -1/2$, where $\chi_t^{(i)}(d) = \left(\kappa(d)T^{(d+1/2)}\right)^{-1}\delta_t^{(i)}\gamma^{(i)}$, and $t \in [0,1]$. In particular, when $i = 2$,*

$$\tilde{x}_T^{(2)}(t) \Rightarrow c_0^{(2)}(d) + \gamma_c t + B_{d+1}^{\underline{c}_t^{(2)}(d), \overline{c}_t^{(2)}(d)}(t)$$

Having stated Theorem 1, the theoretical justification for the inclusion of assumption (e) can now be summarized in the following lemma.

Lemma 1. *Let $\tilde{x}_t^{(2)}$ be a trending RFI(d) process satisfying conditions of Theorem 1. Observe the following statistic:*

$$R = \left(\kappa^2(d)T^{2(d+1/2)}\right)^{-1/2} \left(\max_{t=1\ldots,T} \tilde{x}_t^{(2)} - \min_{t=1\ldots,T} \tilde{x}_t^{(2)}\right)$$

and assume that $\gamma_1 = \gamma_c T^r$. Then, (a) if $r > d - 1/2$, $R \Rightarrow \infty$; (b) if $r = d - 1/2$, $R \Rightarrow \gamma_c \epsilon + \overline{c}(d) - \underline{c}(d)$, for some $0 < \epsilon \leq 1$; (c) if $r < d - 1/2$, $R \Rightarrow \overline{c}(d) - \underline{c}(d)$.

The three cases in Lemma 1 correspond respectively to the three scenarios that would be observed in the bounds of the regulated series. In scenario (a), the bounds would trend away from the series making them useless. Alternatively, in scenario (c), the trend would completely disappear in the limit. It is only in scenario (b) where the bounds persist and the series remains bounded. In other words, the trending regulated series is indeed regulated by said bounds!

3. Variance Ratio Test

The variance ratio test statistic of Nielsen (2009) is a family of non-parametric tests of the unit root hypothesis indexed by the fractional parameter d. To make the exposition clearer, what follows is a brief review of the test statistic in the usual case of unbounded series.

Consider a series $\{y_t\}_{t=1}^T$ and classical unit root hypothesis:

$$y_t = \phi y_{t-1} + u_t, \quad y_0 = 0$$
$$H_0: \ \phi = 1 \quad H_1: \ |\phi| < 1$$

Consider next the series $\tilde{y}_t = \triangle_+^{-d} y_t$. Let σ^2 denote the long-run variance of y_t and denote by $B(t)$ the standard Brownian motion. The variance ratio statistic indexed by d and it's asymptotic distribution under the null hypothesis above is then summarized by the following:

$$\rho(d) = T^{2d} \frac{\sum_{t=1}^{T} y_t^2}{\sum_{t=1}^{T} \tilde{y}_t^2}$$
$$\Rightarrow \frac{\int_0^1 B^2(s)ds}{\frac{1}{\Gamma^2(d+1)} \int_0^1 B_{d+1}^2(s)ds}$$

The statistic $\rho(d)$ above is a generalization of the standard variance ratio test statistic and has the very desirable property of not requiring the estimation of the long-run variance and therefore the serial correlation parameters. Furthermore, it's asymptotic distribution is indexed by d and therefore this indexing parameter is not considered to be a tuning parameter. More importantly, as argued in Müller (2008), these types of statistics consistently discriminate between a null hypothesis of a unit root and an alternative hypothesis of stationarity. For further discussion on the desirable properties of these statistics see Nielsen (2009).

3.1. Regulated Variance Ratio Statistics

As mentioned at the outset of this paper, the objective is to adapt the results of the preceding subsection to regulated series. In this regard, define the series z_t and x_t as follows:

$$z_t = z_{t-1} + u_t \tag{9}$$
$$x_t^{(i)} = \delta_t^{(i)} \gamma^{(i)} + z_t \tag{10}$$

With u_t defined in equation (2), the modified regulator expressions are provided below:

$$\underline{\xi}_t > 0 \quad \text{iff} \quad z_{t-1} + v_t < \underline{b} - \delta_t^{(i)} \gamma^{(i)} \tag{11}$$
$$\overline{\xi}_t > 0 \quad \text{iff} \quad z_{t-1} + v_t > \overline{b} - \delta_t^{(i)} \gamma^{(i)} \tag{12}$$

Observe that when $i = 1$, $x_t^{(1)}$ is the regulated I(1) series studied by Cavaliere and Xu (2011). Under similar assumptions to those found in Assumption 1 of this paper, they show that the cadlag approximation $x_T^{(1)}(t)$ of $x_t^{(1)}$, where,

$$x_T^{(1)}(t) = \left((b_\psi \sigma)^2 T \right)^{-1/2} \left(x_{[Tt]}^{(1)} - x_0^{(1)} \right), \quad t \in [0, 1]$$

tends weakly to the regulated Brownian motion $B^{\underline{c}^1, \overline{c}^1}(t)$. The following theorem generalizes their result to cases when $i = 1, 2$.

Theorem 2. *Let $x_t^{(i)}$ be the regulated I(1) process defined in equations (9) through (12), and fix the bounds $[\underline{b}, \overline{b}]$. If $\mathcal{D}[0, 1]$ is endowed with the Skorohod topology, for all $i = 0, 1, 2$, if $x_0^{(i)} \in [\underline{b}_t^{(i)}, \overline{b}_t^{(i)}]$, then as $T \to \infty$, $x_T^{(i)}(t) \Rightarrow \chi_t^{(i)} + B^{\underline{c}_t^{(i)}, \overline{c}_t^{(i)}}(t)$.*

Now, for $i = 1, 2$, define the sample variances of the regulated series $x_t^{(i)}$ and $\tilde{x}_t^{(i)}$ respectively as:

$$T^{-2} \sum_{t=1}^{T} \left(x_t^{(i)} \right)^2 \Rightarrow (b_\psi \sigma)^2 \int_0^1 B^{\underline{c}^{(i)}, \overline{c}^{(i)}}(s)^2 ds \tag{13}$$

$$T^{-2(1+d)} \sum_{t=1}^{T} \left(\tilde{x}_t^{(i)} \right)^2 \Rightarrow \left(\frac{b_\psi \sigma}{\Gamma(d+1)} \right)^2 \int_0^1 B_{d+1}^{\underline{c}^{(i)}(d), \overline{c}^{(i)}(d)}(s)^2 ds \tag{14}$$

The regulated variance ratio (RVR) statistic can now be defined as follows.

$$\rho^{\underline{c}^{(i)}, \overline{c}^{(i)}}(d) = T^{2d} \frac{\sum_{t=1}^{T} \left(x_t^{(i)} \right)^2}{\sum_{t=1}^{T} \left(\tilde{x}_t^{(i)} \right)^2}, \quad \text{for } i = 1, 2 \tag{15}$$

Moreover, using equations (14) and (15), we see that the limiting distribution of the RVR statistic is given by:

$$\rho^{\underline{c}^{(i)}, \overline{c}^{(i)}}(d) \Rightarrow \Gamma(d+1)^2 \frac{\int_0^1 B^{\underline{c}^{(i)}, \overline{c}^{(i)}}(s)^2 ds}{\int_0^1 B_{d+1}^{\underline{c}^{(i)}(d), \overline{c}^{(i)}(d)}(s)^2 ds}$$

3.2. OLS Detrending

Thus far the $\delta_t^{(i)} \gamma^{(i)}$ term in equations (4) and (9) was treated as if the parameters $\gamma^{(i)}$ were known. Clearly this is rarely the case. In practice one overcomes this through estimation and some detrending procedure. This section will focus on the simplest such procedure through ordinary least squares (OLS) estimation.

For $i = 1, 2$, let $\hat{\gamma}^{(i)}$ denote the OLS estimator of $\gamma^{(i)}$ from regression (4) or (10). The residuals from said regressions can then be expressed as:

$$\hat{x}_t^{(i)} = x_t^{(i)} - \delta_t^{(i)} \hat{\gamma}^{(i)} \tag{16}$$

$$\hat{\tilde{x}}_t^{(i)} = \tilde{x}_t^{(i)} - \delta_t^{(i)} \hat{\gamma}^{(i)} \tag{17}$$

The two equations above say that $\hat{\tilde{x}}_t^{(i)}$ and $\hat{x}_t^{(i)}$ are in fact respectively the detrended versions of $\tilde{x}_t^{(i)}$ and $x_t^{(i)}$. Thus, in accounting for detrending, one must modify the RVR statistic in (15) to use the residuals above. This leads to the version of the RVR statistic presented below.

$$\hat{\rho}^{\underline{c}^{(i)}, \overline{c}^{(i)}}(d) = T^{2d} \frac{\sum_{t=1}^{T} \left(\hat{x}_t^{(i)} \right)^2}{\sum_{t=1}^{T} \left(\hat{\tilde{x}}_t^{(i)} \right)^2}, \quad \text{for } i = 1, 2 \tag{18}$$

The limiting distribution of this static is summarized in the theorem below.

Theorem 3. *Let y_t be defined by equations* (1) *through* (3). *Let $\tilde{x}_t^{(i)}$ be a RFI(d) process defined by equations* (4) *through* (6). *Also, let $x_t^{(i)}$ be a regulated I(1) process defined in equations* (9) *through* (12). *Let the detrended versions of these processes be defined as in* (15) *and* (16). *If $\mathcal{D}[0,1]$ is endowed with the Skorohod topology, $x_0^{(i)}, \tilde{x}_0^{(i)} \in [\underline{b}_t^{(i)}, \overline{b}_t^{(i)}]$, then, for any $d > -1/2$, $i = 1, 2$, and $s \in [0,1]$, as $T \to \infty$,*

$$\hat{\rho}^{\underline{c}^{(i)},\overline{c}^{(i)}}(d) \Rightarrow \frac{\int_0^1 B^{(i)}(t)^2 dt}{\frac{1}{\Gamma^2(d+1)} \int_0^1 B_{d+1}^{(i)}(t)^2 dt}$$

where

$$B^{(i)}(t) = B^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(t)^2 - \delta^{(i)}(t) \left(\int_0^1 \delta^{(i)}(s)^\top \delta^{(i)}(s) ds \right)^{-1} \left(\int_0^1 \delta^{(i)}(s)^\top B^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(s) ds \right)$$

$$B_{d+1}^{(i)}(t) = B_{d+1}^{\underline{c}_t^{(i)}(d),\overline{c}_t^{(i)}(d)}(t)^2 - \delta^{(i)}(t) \left(\int_0^1 \delta^{(i)}(s)^\top \delta^{(i)}(s) ds \right)^{-1} \left(\int_0^1 \delta^{(i)}(s)^\top B_{d+1}^{\underline{c}_t^{(i)}(d),\overline{c}_t^{(i)}(d)}(s) ds \right)$$

where

$$\delta^{(1)}(s) = 1 \quad \textit{and} \quad \delta^{(2)}(s) = [1, s]^\top$$

As observed in Nielsen (2009), variance ratio statistics share a common and very important advantage in the limit. That is, their asymptotic distributions are independent of any short-run dynamics of the innovations u_t. This property continues to hold here in the context of regulated series as well. To see this, take the ratio of the right hand sides of equations (12) and (13) and note that the long-run variance term $(b_\psi \sigma)^2$ disappears and with it any short-run dynamics of u_t which would be captured by b_ψ. In other words, there is no need to perform any estimation on the parameters which capture the short-run dynamics of the innovations or for that matter any long-term variance estimation either. Indexing this family of statistics is therefore very easy considering that asymptotically it only relies on a pre-specification of the fractional integration parameter d and the trend index i.

It is important to note that what drives the asymptotic result of Theorem 3 is in fact assumption (e) above. In the absence of a linear trend, i.e. when $i = 1$, the assumption is not needed and the results obtained in Cavaliere and Xu (2011) and Trokić (2013) can be applied directly. In the presence of a linear trend however, that is when $i = 2$, this assumption is crucial in that it adjusts for the finite sample growth of the trending term so that the series is not absorbed by or diverges from any one of the fixed bounds $[\underline{b}, \overline{b}]$. In other words, the assumption ensures that as the sample size increases, the trending term is always properly scaled so that the series continues to trend and remain within the interval $[\underline{b}, \overline{b}]$.

3.3. Asymptotic Local Power Analysis

Given that the parameter which indexes the RVR family of statistic is d, the question which is of primary concern in this section is whether there exists such a d which maximizes power for said family? To answer this question the same strategy used in Nielsen (2009) will be employed here as well. That is, the power function will be described using local-to-unity asymptotics.

In the presence of near-integrated dynamics, the model considered thus far is modified so that one considers the DGP of $\{y_t\}_{t=1}^{T}$ to be of the form:

$$y_t = \phi_T y_{t-1} + u_t \quad \text{and} \quad \phi_T = 1 - \frac{c}{T} \tag{19}$$

for some $c \geq 0$. In other words, as $T \to \infty$, $\phi_T \to 1$ and the unit root DGP obtains. On the other hand, for any fixed T, the values of $c/T \in (0, 2)$ in (18) imply that y_t is stationary.

It is well known that under nearly integrated alternatives, the limiting distribution of the rescaled sample variance statistics used above follow an Ornstein-Uhlenbeck (O-U) process. In the context of this paper however, such processes need to be modified so as to account for the fact that the series being modelled are bounded and the fact that some series are fractionally integrated. This leads then to the regulated O-U processes, see Theorem 4 in Cavaliere (2005), and the regulated fractional O-U process below:

$$J_c^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(t) = B^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(t) - c\int_0^t e^{-c(t-r)} B^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(r)dr$$

$$J_{d+1,c}^{\underline{c}_t^{(i)}(d),\overline{c}_t^{(i)}(d)}(t) = B_{d+1}^{\underline{c}_t^{(i)}(d),\overline{c}_t^{(i)}(d)}(t) - c\int_0^t e^{-c(t-r)} B_{d+1}^{\underline{c}_t^{(i)}(d),\overline{c}_t^{(i)}(d)}(r)dr$$

This implies that the limiting distribution of the RVR statistic ought to follow a ratio of two O-U processes as is summarized in the following theorem.

Theorem 4. *Let y_t be defined by (18) and u_t through (2) and (3). Let $\tilde{x}_t^{(i)}$ be a RFI(d) process defined by equations (4) through (6). Also, let x_t be a regulated I(1) process defined in equations (9) through (11). Define the detrended versions of these processes as in (15) and (16). If $\mathcal{D}[0,1]$ is endowed with the Skorohod topology, $x_0, \tilde{x}_0^{(i)} \in [\underline{b}_t^{(i)}, \overline{b}_t^{(i)}]$, then, for any $d > -1/2$ and $i = 1, 2$, as $T \to \infty$,*

$$\hat{\rho}_c^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(d) \Rightarrow \frac{\int_0^1 J_c^{(i)}(t)^2 dt}{\frac{1}{\Gamma^2(d+1)}\int_0^1 J_{d+1,c}^{(i)}(t)^2 dt}$$

where

$$J_c^{(i)}(t) = J_c^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(t)^2 - \delta^{(i)}(t)\left(\int_0^1 \delta^{(i)}(s)^\top \delta^{(i)}(s)ds\right)^{-1}\left(\int_0^1 \delta^{(i)}(s)^\top J_c^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(s)ds\right)$$

$$J_{d+1,c}^{(i)}(t) = J_{d+1,c}^{\underline{c}_t^{(i)}(d),\overline{c}_t^{(i)}(d)}(t)^2 - \delta^{(i)}(t)\left(\int_0^1 \delta^{(i)}(s)^\top \delta^{(i)}(s)ds\right)^{-1}\left(\int_0^1 \delta^{(i)}(s)^\top J_{d+1,c}^{\underline{c}_t^{(i)}(d),\overline{c}_t^{(i)}(d)}(s)ds\right)$$

and

$$\Pi_{1,r} = 1 \quad and \quad \Pi_{2,r} = [1, r]^\top$$

Using simulation analysis, Nielsen (2009) suggested that the "power maximizing" choice of d for the variance ratio test is $d = 0.1$. Here power maximizing is not a maximum in the theoretical sense as choices of $d < 0.1$ yield uniformly (in c) higher power. The danger in choosing d too small however is that d begins to behave as if it is inversely proportional to the sample size resulting in poor size properties of the test. As will be seen in the following section, the case of regulated variance ratio statistics demands a careful selection of the power-maximizing choice of the fractional parameter d.

4. Simulation Analysis

The preceding sections established the theoretical justifications for regulated variance ratio statistic. Here this theory is applied to study the choices of d which maximize power in case of a unit root test for bounded $I(1)$ times series. In this regard, the simulation experiments which follow confirm that indeed, choosing d as small as possible yields uniformly (in c) higher power. Nevertheless, unlike the case of unbounded series, choices of d greater than 0.5 are not recommended as they actually lead to sharp power loss. In other words, testing for the presence of a unit root using the Breitung (2002) test statistic where $d = 1$ while ignoring the bounded nature of a time series would be highly misleading.

Analogous to the unbounded version of the test, local power of the regulated variance ratio statistic is calculated using the following formula:

$$Pr\left\{\hat{\rho}_c^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(d) > q_{1-\alpha}^{(i)}(d)\right\} \tag{20}$$

where $q_{1-\alpha}^{(i)}(d)$ is the $1-\alpha$ quantile of the distribution of $\hat{\rho}^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(d)$. That is,

$$Pr\left\{\hat{\rho}^{\underline{c}_t^{(i)},\overline{c}_t^{(i)}}(d) < q_{1-\alpha}^{(i)}(d)\right\} = 1-\alpha$$

Stated differently, the test rejects the unit root null hypothesis for large values of the test statistic. Critical values for the regulated variance ratio test are tabulated in Table 1 for various configurations.

Bounds	α	T	$d = 0.10$	$d = 0.25$	$d = 0.50$	$d = 1.00$
$[0, 1]$	0.10	100	1.495	2.176	3.176	6.529
		500	1.565	2.175	2.956	5.818
	0.05	100	1.523	2.261	3.358	7.031
		500	1.581	2.214	3.027	5.987
	0.01	100	1.576	2.441	3.745	8.130
		500	1.613	2.292	3.164	6.301
$[0, 10]$	0.10	100	1.350	1.947	3.279	9.473
		500	1.319	1.781	2.639	6.003
	0.05	100	1.395	2.083	3.697	11.623
		500	1.347	1.856	2.812	6.703
	0.01	100	1.481	2.384	4.624	17.457
		500	1.403	2.008	3.206	8.324
$[0, 100]$	0.10	100	1.354	1.971	3.393	10.561
		500	1.322	1.863	3.187	9.809
	0.05	100	1.397	2.107	3.849	13.135
		500	1.356	1.985	3.547	11.788
	0.01	100	1.483	2.410	4.880	19.050
		500	1.424	2.222	4.340	16.846

Table 1: Critical values of the regulated variance ratio test statistics for the case of no trend and no drift. The results were obtained over 25,000 Monte Carlo replications.

Consider now the finite sample performance in terms of local power of the regulated variance ratio statistic when the time series is regulated.[2] In particular, the time series y_t is simulated using the following data generating process:

$$y_t = y_{t-1} + u_t, \quad y_0 = 0$$

where

$$\begin{aligned} \underline{\xi}_t > 0 \quad &\text{iff} \quad y_{t-1} + v_t < \underline{b} \\ \overline{\xi}_t > 0 \quad &\text{iff} \quad y_{t-1} + v_t > \overline{b} \end{aligned}$$

for various configurations of the bounds $\underline{b}$ and $\overline{b}$. In particular, consider first the case of time series with bounds to the right of zero. Such processes are of enormous importance to empirical economists, especially those dealing with macroeconomic time series. In

[2]All simulation experiments were conducted by defining distributional bounds with respect to the standard regulated time series, that is the interval characterized by $[\underline{c}_t^{(i)}, \overline{c}_t^{(i)}]$. Furthermore, simulations were conducted over 25,000 Monte Carlo replications. Here, it should also be noted that there are several functional forms available for the regulator functions $\underline{\xi}_t$ and $\overline{\xi}_t$. Cavaliere (2005) mentions several ways to define ξ in the case of standard regulated series whereas Trokić (2013) presents a discussion on these variants in the case of regulated fractionally integrated series. Simulations conducted in this paper only consider the case of absorbing bounds.

fact, a number of important economic series are bounded below by zero and above by some number. A salient example of such a series is the unemployment rate which is always non-negative and never greater than 100. Depending on the particular coding, such a series may be bounded between 0 and 1 or between 0 and 100. Consider Figures 1 and 2 which illustrate the local power curves of the regulated fractional variance ratio statistic for various values of d for both coding scenarios.

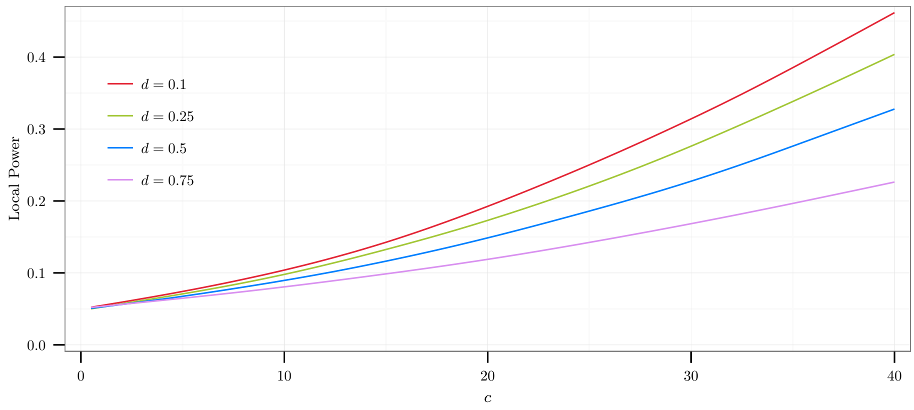

Figure 1: Local power for a series of length $T = 100$ and bounded between $[\underline{b}, \overline{b}] = [0, 1]$.

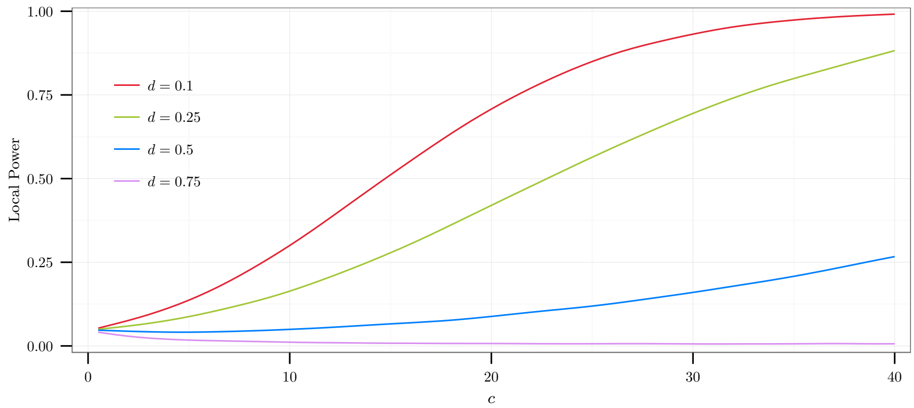

Figure 2: Local power for a series of length $T = 100$ and bounded between $[\underline{b}, \overline{b}] = [0, 100]$.

Although it is clear that local power increases for choices of d less than unity, the presence of bounds significantly reduces power of the fractional variance ratio test. Furthermore, Figure 1 seems to suggest that for very tight bounds, choices of d less than $1/2$ tend to perform better than in cases when the bounds are relaxed. This illusory effect is masking the fact that as the bounds become tighter, the velocity with which the bounded integrated series reaches the bounds is increasing. In particular, although tighter bounds imply less nonstationarity in the series, the increased interaction with the bounds creates an ARCH-like effect in the process, rendering the series seemingly more nonstationary. On the other hand, as the upper bound begins to widen, choices of $d < 1/2$ dominate choices of $d > 1/2$ as the former increasingly force the test statistic $\hat{\rho}^{\underline{c}_t^{(i)}, \overline{c}_t^{(i)}}(d)$ to behave more like a constant when the lower bound $\underline{b} = 0$ is reached. The conclusion also seems to be independent of whether symmetric or asymmetric bounds are present. In this regard, Figure 3 which illustrates local power for a series bounded by the symmetric interval $[\underline{b}, \overline{b}] = [-1, 1]$, is not markedly different from Figure 1 which illustrates the same for bounds $[\underline{b}, \overline{b}] = [0, 1]$. This seems to indicate that the direction and symmetry of the bounds are not nearly as significant as the distance between the bounds. This is further supported in Figure 4 which presents local asymptotic power curves for fixed $d = 0.1$ and various bound configurations. It goes without saying that distancing the lower and upper bounds far enough from zero recovers the asymptotic power curves of Nielsen (2009).

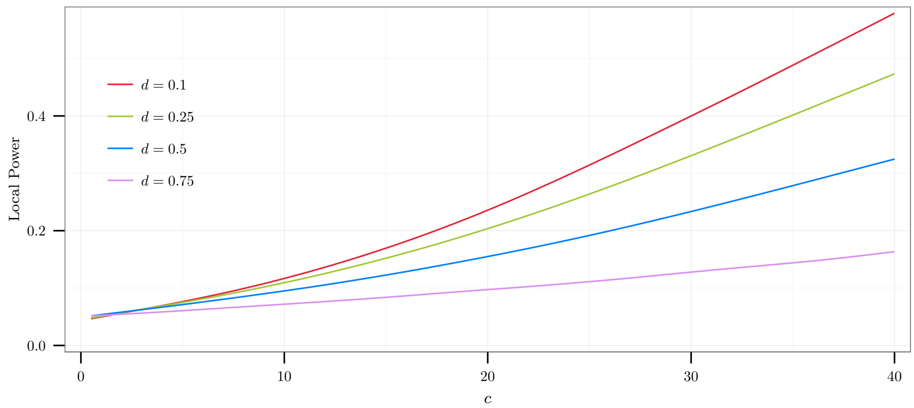

Figure 3: Local asymptotic power for a series of length $T = 100$ and bounded between $[\underline{b}, \overline{b}] = [-1, 1]$.

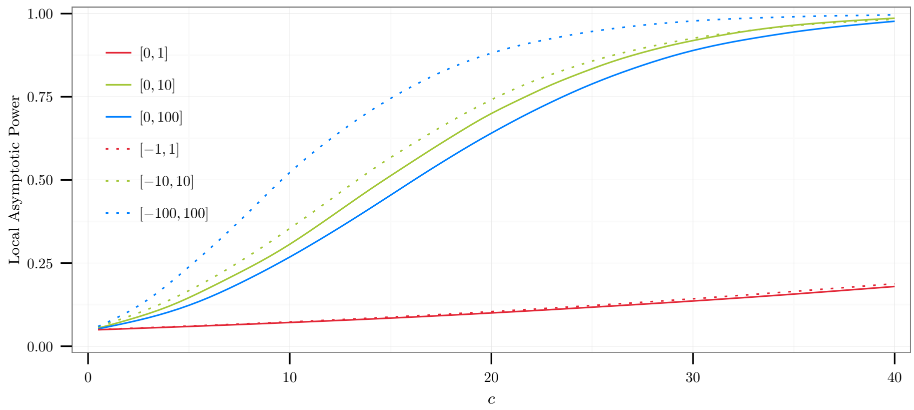

Figure 4: Local asymptotic power for a series of length $T = 500$ with $d = 0.1$.

5. Conclusion

Regulated time series are of tremendous practical importance. They characterize a number of important time series and are particularly salient in macroeconomic literature. As demonstrated in Cavaliere (2005), Cavaliere and Xu (2011), and Trokić (2013), these series have asymptotic distributions which are quite different than those characterizing unbounded series. Unfortunately, the unit root literature has only recently begun to take this into consideration. This article contributes in this direction by extending the fractional variance ratio statistic of Nielsen (2009) to regulated $I(1)$ time series. This extension relies on the theory of regulated fractionally integrated series developed in Trokić (2013) which is used here to develop the regulated fractional variance ratio statistic along with its limiting distribution. Furthermore, this article proposes a theoretical justification (based on local linear trends) for limiting distributions of regulated series with a linear trend. The latter was developed for fractionally integrated series of general order $d > -1/2$. Simulation evidence for local (asymptotic) power curves confirms the results of Nielsen (2009) and illustrates that practitioners should exercise caution when applying the fractional variance ratio statistic to series which may be regulated.

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Appendix

Proof of Lemma 1. Note that the RFI(d) process considered in equations (1) through (6) with $i = 2$ can always be expressed as

$$\tilde{x}_t^{(2)} = \tilde{x}_t^{(1)} + \gamma_1 t$$

This implies that the corresponding process in $\mathcal{D}[0,1]$ is in fact:

$$\tilde{x}_{[Tt]}^{(2)} = \tilde{x}_{[Tt]}^{(1)} + \gamma_1 [Tt]$$

Impose now the constraint that $\gamma_1 = \gamma_c T^r$ and note that the range statistic R can be expressed as follows:

$$\begin{aligned}
R &= \left(\kappa^2(d) T^{2(d+1/2)}\right)^{-1/2} \left(\sup_{t\in[0,1]} \tilde{x}_{[Tt]}^{(2)} - \inf_{t\in[0,1]} \tilde{x}_{[Tt]}^{(2)} \right) \\
&= \sup_{t\in[0,1]} \left(\kappa^2(d) T^{2(d+1/2)}\right)^{-1/2} \left(\tilde{x}_t^{(1)} + \gamma_c T^r [Tt] \right) \\
&- \inf_{t\in[0,1]} \left(\kappa^2(d) T^{2(d+1/2)}\right)^{-1/2} \left(\tilde{x}_t^{(1)} + \gamma_c T^r [Tt] \right) \\
&= \sup_{t\in[0,1]} \left(\left(\kappa^2(d) T^{2(d+1/2)}\right)^{-1/2} \tilde{x}_t^{(1)} + \frac{\gamma_c}{\kappa(d)} [T^{r+1/2-d} t] \right) \\
&- \inf_{t\in[0,1]} \left(\left(\kappa^2(d) T^{2(d+1/2)}\right)^{-1/2} \tilde{x}_t^{(1)} + \frac{\gamma_c}{\kappa(d)} [T^{r+1/2-d} t] \right)
\end{aligned}$$

Note however that $\left(\kappa^2(d) T^{2(d+1/2)}\right)^{-1/2} \tilde{x}_t^{(1)} \Rightarrow B_d^{\underline{c}^{(1)}(d),\overline{c}^{(1)}(d)}(t)$ for $t \in [0,1]$ by standard arguments in Trokić (2013), or for that matter Theorem 1. Suppose now that the process $\tilde{x}_{[Tt]}^{(2)}$ achieves its infimum at some t_{inf} and its supremum at some $t_{\text{sup}} = t_{\text{inf}} + \epsilon$ for some $0 < \epsilon \leq 1$. Then, the above implies that:

$$\begin{aligned}
R &\Rightarrow \lim_{T\to\infty} \left(\frac{\gamma_c}{\kappa(d)} [T^{r+1/2-d} t_{\text{sup}}] \right) - \lim_{T\to\infty} \left(\frac{\gamma_c}{\kappa(d)} [T^{r+1/2-d} t_{\text{inf}}] \right) + B_d^{\underline{c}^{(1)}(d),\overline{c}^{(1)}(d)}(t_{\text{sup}}) - B_d^{\underline{c}^{(1)}(d),\overline{c}^{(1)}(d)}(t_{\text{inf}}) \\
&\Rightarrow \lim_{T\to\infty} \left(\frac{\gamma_c}{\kappa(d)} [T^{r+1/2-d} t_{\text{sup}}] \right) - \lim_{T\to\infty} \left(\frac{\gamma_c}{\kappa(d)} [T^{r+1/2-d} t_{\text{inf}}] \right) + \overline{c}^{(1)}(d) - \underline{c}^{(1)}(d)
\end{aligned}$$

Consider now the three cases in Lemma 1. When $r > d - 1/2$, the first two terms in the two expressions above explode and $R \Rightarrow \infty$. On the other hand, when $r < d - 1/2$, those same two terms approach zero and $R \Rightarrow \overline{c}(d) - \underline{c}(d)$ since $\overline{c}^{(1)}(d) = \overline{c}(d) + \gamma_0$ and $\underline{c}^{(1)}(d) = \underline{c}(d) + \gamma_0$. This implies that as the sample size grows, the trend completely disappears and the originally trending regulated series is now indistinguishable from a regulated series without a trend. It is only in the last case when $r = d - 1/2$ that the trend persists throughout the lifetime of the series. In this case,

$$R \Rightarrow \frac{\gamma_c}{\kappa(d)}\left(t_{\sup} - t_{\inf}\right) + \overline{c}(d) - \underline{c}(d)$$
$$\Rightarrow \frac{\gamma_c}{\kappa(d)}\epsilon + \overline{c}(d) - \underline{c}(d)$$

In other words, the range of the series is always the range of $[\underline{c(d)}, \overline{c(d)}]$ plus the term arising from the trend, namely $\frac{\gamma_c}{\kappa(d)}\epsilon$. Thus, when the infimum and supremum points of the series are at 1 and 0 respectively, the range of the statistic is indeed, $\frac{\gamma_c}{\kappa(d)} + \overline{c}(d) - \underline{c}(d)$. Put differently, the range between any two points in the series, arbitrarily close to one another, is always bounded above by something which can be made arbitrarily close to $\overline{c}(d) - \underline{c}(d)$. □

Proof of Theorem 1. For the case when $i = 0, 1$ the proof is identical to that found in Trokić (2013). Fortunately, when $i = 2$, the proof is similar. Consider therefore:

$$\tilde{\tilde{x}}_t^{(2)} = \begin{cases} \overline{b}_t^{(2)} & \text{if } \tilde{\tilde{x}}_{t-1}^{(2)} + \gamma_1 + \triangle_+^{-d} v_t > \overline{b}_t^{(2)} \\ \underline{b}_t^{(2)} & \text{if } \tilde{\tilde{x}}_{t-1}^{(2)} + \gamma_1 + \triangle_+^{-d} v_t < \underline{b}_t^{(2)} \\ \tilde{\tilde{x}}_{t-1}^{(2)} + \gamma_1 + \triangle_+^{-d} v_t & \text{otherwise} \end{cases}$$

and assume that $\tilde{\tilde{x}}_0^{(2)} = \gamma_0$. This implies that the process can be recursively defined as:

$$\begin{aligned} \tilde{\tilde{X}}_t^{(2)} &= \tilde{\tilde{x}}_0^{(2)} + \sum_{i=1}^{t} \triangle_+^{-d} v_i + \sum_{i=1}^{t} \triangle_+^{-d} \underline{\xi}_{d,i} - \sum_{i=1}^{t} \triangle_+^{-d} \overline{\xi}_{d,i} \\ &= \gamma_0 + \gamma_1 t + V_t + L_t - M_t \end{aligned}$$

Note that the above sequence is in fact the Harrison (1985) construction of a regulated stochastic process but it is not continuous. Thus, if we can obtain a continuous approximation of the above sequence on the $\mathcal{C}[0,1]$ space with a uniform metric and show convergence, convergence in the original $\mathcal{D}[0,1]$ space will follow by Theorem 4.1 in Billingsley (1968). To this end, one can define the regulators as follows:

$$\begin{aligned} \triangle_+^{-d} \underline{\xi}_{d,t} = \underline{\xi}_t &= \max\left(0, \underline{b}_t^{(2)} - \tilde{\tilde{X}}_{t-1}^{(2)} - \gamma_1 - \triangle_+^{-d} v_t\right) \\ \triangle_+^{-d} \overline{\xi}_{d,t} = \overline{\xi}_t &= \max\left(0, \tilde{\tilde{X}}_{t-1}^{(2)} + \gamma_1 + \triangle_+^{-d} v_t - \overline{b}_t^{(2)}\right) \end{aligned}$$

and clearly $\tilde{\tilde{X}}_t^{(2)} \in [\underline{b}_t^{(2)}, \overline{b}_t^{(2)}]$.

Next, apply the broken line process to $\widetilde{\widetilde{X}}_t^{(2)}$ and create its continuous approximant on $\mathcal{C}[0,1]$ with $t \in [0,1]$ as follows:

$$\widetilde{\widetilde{X}}_T^{(2)}(t) = \left(\kappa^2(d) T^{2(d+1/2)}\right)^{-1/2} \widetilde{\widetilde{X}}_{[Tt]}^{(2)}$$

Do the same for the other terms:

$$V_T^*(t) = \left(\kappa^2(d) T^{2(d+1/2)}\right)^{-1/2} \sum_{i=1}^{[Tt]} \triangle_+^{-d} v_i + \triangle_+^{-d} v_{[Tt]+1} \left(\frac{Tt - [Tt]}{\kappa(d) T^{(d+1/2)}}\right)$$

$$L_T^*(t) = \begin{cases} \left(\kappa^2(d)T^{2(d+1/2)}\right)^{-1/2} \sum_{i=1}^{[Tt]} \triangle_+^{-d} \underline{\xi}_{d,i} & \text{if} \quad \triangle_+^{-d}\underline{\xi}_{d,[Tt]+1} = 0 \\ \\ \left(\kappa^2(d)T^{2(d+1/2)}\right)^{-1/2} \sum_{i=1}^{[Tt]} \triangle_+^{-d} \underline{\xi}_{d,i} & \text{if} \quad \triangle_+^{-d}\underline{\xi}_{d,[Tt]+1} > 0 \\ + \left(\kappa^2(d)T^{2(d+1/2)}\right)^{-1/2} \left(\frac{Tt - [Tt] - \frac{v_{[Tt]+1} - \triangle_+^{-d}\underline{\xi}_{d,[Tt]+1}}{\triangle_+^{-d}\underline{\xi}_{d,[Tt]+1}}}{1 - \frac{v_{[Tt]+1} - \triangle_+^{-d}\underline{\xi}_{d,[Tt]+1}}{\triangle_+^{-d}\underline{\xi}_{d,[Tt]+1}}} \right) \triangle_+^{-d}\underline{\xi}_{d,[Tt]+1} & \\ \times \mathbb{I}_{\left\{ Tt \geq [Tt] + \frac{v_{[Tt]+1} - \triangle_+^{-d}\underline{\xi}_{d,[Tt]+1}}{\triangle_+^{-d}\underline{\xi}_{d,[Tt]+1}} \right\}} & \end{cases}$$

$$M_T^*(t) = \begin{cases} \left(\kappa^2(d)T^{2(d+1/2)}\right)^{-1/2} \sum_{i=1}^{[Tt]} \triangle_+^{-d} \overline{\xi}_{d,i} & \text{if} \quad \triangle_+^{-d}\overline{\xi}_{d,[Tt]+1} = 0 \\ \\ \left(\kappa^2(d)T^{2(d+1/2)}\right)^{-1/2} \sum_{i=1}^{[Tt]} \triangle_+^{-d} \overline{\xi}_{d,i} & \text{if} \quad \triangle_+^{-d}\overline{\xi}_{d,[Tt]+1} > 0 \\ + \left(\kappa^2(d)T^{2(d+1/2)}\right)^{-1/2} \left(\frac{Tt - [Tt] - \frac{v_{[Tt]+1} - \triangle_+^{-d}\overline{\xi}_{d,[Tt]+1}}{\triangle_+^{-d}\overline{\xi}_{d,[Tt]+1}}}{1 - \frac{v_{[Tt]+1} - \triangle_+^{-d}\overline{\xi}_{d,[Tt]+1}}{\triangle_+^{-d}\overline{\xi}_{d,[Tt]+1}}} \right) \triangle_+^{-d}\overline{\xi}_{d,[Tt]+1} & \\ \times \mathbb{I}_{\left\{ Tt \geq [Tt] + \frac{v_{[Tt]+1} - \triangle_+^{-d}\overline{\xi}_{d,[Tt]+1}}{\triangle_+^{-d}\overline{\xi}_{d,[Tt]+1}} \right\}} & \end{cases}$$

The above implies that

$$\widetilde{\widetilde{X}}_T^{*(2)}(t) = \left(\kappa(d) T^{(d+1/2)}\right)^{-1} \gamma_0 + \left(\kappa(d) T^{(d+1/2)}\right)^{-1} \gamma_1 [Tt] + V_T^*(t) + L_T^*(t) - M_T^*(t)$$

Invoking now assumption (e), that is, setting $\gamma_1 = \gamma_c T^{d-1/2}$, the following expression obtains:

$$\widetilde{\widetilde{X}}_T^{*(2)}(t) = \left(\kappa(d)T^{(d+1/2)}\right)^{-1}\gamma_0 + \kappa(d)^{-1}\gamma_1 t + V_T^*(t) + L_T^*(t) - M_T^*(t)$$

Notice that by the results in Wang et al. (2002), $V_T(t) \Rightarrow B_{d+1}(t)$, in other words, a type II fractional Brownian motion with parameter d. What needs to be demonstrated for the proof to follow however, is that $V_T^*(t) \Rightarrow B_{d+1}(t)$. This obtains by noting that

$$\begin{aligned}
\sup_{t\in[0,1]} |V_T^*(t) - V_T(t)| &= \sup_{t\in[0,1]} \left| \triangle_+^{-d} v_{[Tt]+1} \left(\frac{Tt - [Tt]}{\kappa(d)T^{(d+1/2)}} \right) \right| \\
&\leq \left(\kappa(d)T^{(d+1/2)}\right)^{-1} \max_{t=1,\ldots,T} \left|\triangle_+^{-d} v_t\right| \\
&= o_P(1)
\end{aligned}$$

The above inequality follows from the second part of assumption (b), a result by Gourieroux and Akonom (1988) and the first part of the proof of Theorem 1 in Wang et al. (2002). Invoking Theorem 4.1 of Billingsley (1968) then implies that $V_T^*(t) \Rightarrow B_{d+1}(t)$.

The main part of the proof now follows by noting that $\widetilde{\widetilde{X}}_T^{*(2)}(t)$ obtained above satisfies the Harrison (1985) construction for regulated series. In particular, his Proposition 2.4.6 claims that $\widetilde{\widetilde{X}}_T^{*(2)}(t)$ is the unique functional which regulates

$$\left(\kappa(d)T^{(d+1/2)}\right)^{-1}\gamma_0 + \kappa(d)^{-1}\gamma_1 t + V_T^*(t)$$

By assumption (f), this implies that in the limit, the above lies the interval $[\underline{c}_t^{(2)}(d), \overline{c}_t^{(2)}(d)]$. The continuing mapping theorem (CMT) and the fact that $V_T^*(t) \Rightarrow B_{d+1}(t)$ then entails that

$$\begin{aligned}
\widetilde{\widetilde{X}}_T^{*(2)}(t) &\Rightarrow g^{\underline{c}_t^{(2)}(d),\overline{c}_t^{(2)}(d)} \left(\left(\kappa(d)T^{(d+1/2)}\right)^{-1}\gamma_0 + \kappa(d)^{-1}\gamma_1 t + B_{d+1}(t) \right) \\
&= g^{\underline{c}_t^{(2)}(d),\overline{c}_t^{(2)}(d)} \left(\chi_t^{(2)} + B_{d+1}(t) \right)
\end{aligned}$$

where $g^{\underline{c}_t^{(2)}(d),\overline{c}_t^{(2)}(d)}(\cdot)$ is the regulating function which constrains everything to the appropriate interval. What remains to be shown is that $\widetilde{\widetilde{X}}_T^{*(2)}(t)$ converges to $\widetilde{\widetilde{X}}_T^{(2)}(t)$. A first step toward this result is to note the following:

$$
\begin{aligned}
&\left|\widetilde{\widetilde{X}}_T^{*(2)}(t) - \widetilde{\widetilde{X}}_T^{(2)}(t)\right| \\
&= \left|\left(L_T^*(t) - \frac{L_{[Tt]}}{\kappa(d)T^{(d+1/2)}}\right) - \left(M_T^*(t) - \frac{M_{[Tt]}}{\kappa(d)T^{(d+1/2)}}\right) + \triangle_+^{-d} v_{[Tt]+1}\left(\frac{Tt-[Tt]}{\kappa(d)T^{(d+1/2)}}\right)\right| \\
&\leq \frac{2}{\kappa(d)T^{(d+1/2)}}\left|\triangle_+^{-d} v_{[Tt]+1}\right|
\end{aligned}
$$

The inequality above holds since both $\left|L_T^*(t) - \frac{L_{[Tt]}}{\kappa(d)T^{(d+1/2)}}\right|$ and $\left|M_T^*(t) - \frac{M_{[Tt]}}{\kappa(d)T^{(d+1/2)}}\right|$ are smaller than $\left(\kappa(d)T^{(d+1/2)}\right)^{-1}\left|\triangle_+^{-d} v_{[Tt]+1}\right|$ and due to the fact that the nonzero set of $L_T^*(t)$ and $M_T^*(t)$ are clearly disjoint. Convergence follows by the same arguments used in the convergence of $S_T^*(t)$. That is,

$$
\sup_{t\in[0,1]}\left|\widetilde{\widetilde{X}}_T^{*(2)}(t) - \widetilde{\widetilde{X}}_T^{(2)}(t)\right| \leq \frac{2}{\kappa(d)T^{(d+1/2)}}\left|\triangle_+^{-d} v_t\right| = o_P(1)
$$

This completes the part of the proof which demonstrates that $\widetilde{\widetilde{X}}_T^{(2)}(t) \Rightarrow \chi_t^{(2)} + B_{d+1}^{\underline{c}_t^{(2)}(d),\overline{c}_t^{(2)}(d)}(t)$. To prove the postulate of Theorem 1 however, the convergence of $\widetilde{\widetilde{X}}_T^{(2)}(t)$ to $\widetilde{X}_T^{(2)}(t)$ remains to be shown. This is completed in Lemma 2. □

Lemma 2. *Let* $\left\{\widetilde{\widetilde{X}}_t^{(i)}\right\}$ *and* $\left\{\widetilde{X}_t^{(i)}\right\}$ *be defined as above for any i=0,1,2. Then,*

$$
\begin{aligned}
\max_{t=0,\ldots,T}\left|\widetilde{\widetilde{X}}_t^{(i)} - \widetilde{X}_t^{(i)}\right| &\leq \max\left(\max_{t=0,\ldots,T}\triangle_+^{-d}\underline{\xi}_{d,t}, \max_{t=0,\ldots,T}\triangle_+^{-d}\overline{\xi}_{d,t}\right) \\
&= \max\left(\max_{t=0,\ldots,T}\triangle_+^{-d}\underline{\xi}_{d,t}, \max_{t=0,\ldots,T}\triangle_+^{-d}\overline{\xi}_{d,t}\right) \\
&\leq \max_{t=0,\ldots,T}\triangle_+^{-d}\underline{\xi}_{d,t} + \max_{t=0,\ldots,T}\triangle_+^{-d}\overline{\xi}_{d,t}
\end{aligned}
$$

Proof of Lemma 2. This is in fact just a version of Lemma 7 in Cavaliere (2005) and the proof there remains valid here. The convergence of $\widetilde{\widetilde{X}}_T^{(2)}(t)$ to $\widetilde{X}_T^{(2)}(t)$ follows by recalling assumption (d), the result by Gourieroux and Akonom (1988), and the first part of the proof of Theorem 1 in Wang et al. (2002) which can be used to show that $\max\triangle_+^{-d}\xi_{d,t}$ is $o_P(T^{d+1/2})$ for both the lower and upper regulator. This fact then implies that

$$
\left|\widetilde{\widetilde{X}}_T^{(2)}(t) - \widetilde{X}_T^{(2)}(t)\right| \Rightarrow 0
$$

and the proof of Theorem 1 is complete. □

PROOF OF THEOREM 2. The proof of Theorem 2 for the cases $i = 0, 1$ were covered in Cavaliere (2005). The case for $i = 2$ follows virtually the same line of argumentation as that found the proof of Theorem 1 above with a few minor alterations.

The changes which need to be introduced on the assumptions are indexed by double letters. These assumptions are very close to those presented in Cavaliere (2005) and are as follows.

(a) $\{\epsilon_t, \mathcal{F}_t\}$ is a MDS with respect to some filtration $\mathcal{F}_t$ and $E\{\epsilon_t^2|\mathcal{F}_t\} = \sigma^2 < \infty$.

(bb) $\sup_{t\in\mathbf{Z}} E\{|\epsilon_t|^p\} < \infty$ for $p = 2 + \eta$ for some $\eta > 0$.

(c) $\sum_{j=0}^{\infty} |\psi_j| < \infty$, $\sum_{j=0}^{\infty} j|\psi_j| < \infty$, and $b_\psi = \sum_{j=0}^{\infty} \psi_j \neq 0$.

(dd) $\sup_{t\in\mathbf{Z}} E\{|\underline{\xi}_t|^p\} < \infty$ and $\sup_{t\in\mathbf{Z}} E\{|\bar{\xi}_t|^p\} < \infty$, for p as in (b).

(ee) $\gamma_1 = \gamma_c T^r$ where γ_c is a fixed constant and $r = -1/2$.

(ff) $\left(\kappa T^{1/2}\right)^{-1} \underline{b}_t^{(i)} = \underline{c}_t^{(i)} + o(1)$

$\left(\kappa T^{1/2}\right)^{-1} \bar{b}_t^{(i)} = \bar{c}_t^{(i)} + o(1)$

for $i = 0, 1, 2$, where $\kappa = b_\psi \sigma$ and $\underline{c}^{(i)} \neq \bar{c}^{(i)}$.

The changes which need to be made on the structural side are those which replace $\triangle_+^{-d} v_t$ with just v_t. Then, by noting that by Gourieroux and Akonom (1988), assumptions (bb) and (ff) imply that:

$$\max_{t=0,\dots,T} |\epsilon_t| = o_P(T^{1/2}) \quad \text{and} \quad \sup_{t\in[0,1]} E\left\{\left|\underline{\xi}_t\right|\right\} = o_P(T^{1/2})$$

the convergence of the modified $V_T^*(t)$ to $B^{\underline{c}_t^{(2)},\bar{c}_t^{(2)}}(t)$ follows, as does the convergence of the modified $\tilde{\tilde{X}}_T^{(2)}(t)$ to $\chi_t^{(2)} + B^{\underline{c}_t^{(2)},\bar{c}_t^{(2)}}(t)$. Finally, the convergence of $\tilde{\tilde{X}}_T^{(2)}(t)$ to $X_T^{(2)}(t)$ follows by Lemma 2 above with invocations of the modified assumptions found above. This completes the proof of Theorem 2. □

PROOF OF THEOREM 3. Begin with the non-fractional case and consider the $\mathcal{D}[0,1]$ approximation of the residuals from the regression of $x_t^{(i)}$ on $\delta_t^{(i)}\gamma^{(i)}$ for $t = 1 \dots T$, by noting the following for $s \in [0,1]$:

$$\begin{aligned}\hat{x}_{\lfloor Ts\rfloor}^{(i)} &= x_{\lfloor Ts\rfloor}^{(i)} - \delta_{\lfloor Ts\rfloor}^{(i)}\hat{\gamma}^{(i)} \\ &= z_{\lfloor Ts\rfloor} - \delta_{\lfloor Ts\rfloor}^{(i)}\left(\hat{\gamma}^{(i)} - \gamma^{(i)}\right)\end{aligned}$$

Define $N^{(1)}(T) = 1$ and $N^{(1)}(T) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & T^{-1} \end{smallmatrix}\right]$ and note the following:

$$
\begin{aligned}
&(b_\psi \sigma)^{-1} T^{-1/2} \delta_{\lfloor Ts \rfloor}^{(i)} \left(\hat{\gamma}^{(i)} - \gamma^{(i)} \right) \\
&= N^{(i)}(T) \delta_{\lfloor Ts \rfloor}^{(i)} \left(N^{(i)}(T) \left(b_\psi \sigma \right)^{-1} T^{-1/2} \left(\hat{\gamma}^{(i)} - \gamma^{(i)} \right) \right) \\
&= N^{(i)}(T) \delta_{\lfloor Ts \rfloor}^{(i)} \left(T^{-1} \sum_{t=1}^{T} N^{(i)}(T) \delta_t^{(i)\top} N^{(i)}(T) \delta_t^{(i)} \right)^{-1} \left(T^{-1} \sum_{t=1}^{T} N^{(i)}(T) \delta_t^{(i)\top} \left(b_\psi \sigma \right)^{-1} T^{-1/2} z_{\lfloor Ts \rfloor} \right) \\
&= N^{(i)}(T) \delta_{\lfloor Ts \rfloor}^{(i)} \left(T^{-1} \sum_{t=1}^{T} \delta_{t/T}^{(i)\top} \delta_{t/T}^{(i)} \right)^{-1} \left(T^{-1} \sum_{t=1}^{T} \delta_{t/T}^{(i)\top} \left(b_\psi \sigma \right)^{-1} T^{-1/2} z_{\lfloor Ts \rfloor} \right) \\
&\Rightarrow \delta^{(i)}(s) \left(\int_0^1 \delta^{(i)}(s)^\top \delta^{(i)}(s) ds \right)^{-1} \left(\int_0^1 \delta^{(i)}(s)^\top B^{\underline{c}^{(i)}, \bar{c}^{(i)}}(s) ds \right)
\end{aligned}
$$

The above follows from Theorem 2, the continuous mapping theorem, and from the fact that as $T \to \infty$, $t/T \to r$ and therefore $N^{(i)}(T) \delta_{\lfloor Ts \rfloor}^{(i)} \to \delta^{(i)}(s)$. Thus, conclude that

$$
(b_\psi \sigma)^{-1} T^{-1/2} \hat{x}_{\lfloor Ts \rfloor}^{(i)} \Rightarrow B^{\underline{c}^{(i)}, \bar{c}^{(i)}}(s) - \delta^{(i)}(s) \left(\int_0^1 \delta^{(i)}(r)^\top \delta^{(i)}(r) dr \right)^{-1} \left(\int_0^1 \delta^{(i)}(r)^\top B^{\underline{c}^{(i)}, \bar{c}^{(i)}}(r) dr \right)
$$

A similar argument can be used for the fractional case. Consider therefore the following:

$$
\hat{\tilde{x}}_{\lfloor Ts \rfloor}^{(i)} = \tilde{y}_{\lfloor Ts \rfloor} - \delta_{\lfloor Ts \rfloor}^{(i)} \left(\hat{\gamma}^{(i)} - \gamma^{(i)} \right)
$$

and note that

$$
\begin{aligned}
&\left(\kappa^2(d) T^{2(d+1/2)} \right)^{-1/2} \delta_{\lfloor Ts \rfloor}^{(i)} \left(\hat{\gamma}^{(i)} - \gamma^{(i)} \right) \\
&= N^{(i)}(T) \delta_{\lfloor Ts \rfloor}^{(i)} \left(N^{(i)}(T) \left(\kappa^2(d) T^{2(d+1/2)} \right)^{-1/2} \left(\hat{\gamma}^{(i)} - \gamma^{(i)} \right) \right) \\
&= N^{(i)}(T) \delta_{\lfloor Ts \rfloor}^{(i)} \left(T^{-1} \sum_{t=1}^{T} N^{(i)}(T) \delta_t^{(i)\top} N^{(i)}(T) \delta_t^{(i)} \right)^{-1} \left(T^{-1} \sum_{t=1}^{T} N^{(i)}(T) \delta_t^{(i)\top} \left(\kappa^2(d) T^{2(d+1/2)} \right)^{-1/2} \tilde{z}_{\lfloor Ts \rfloor} \right) \\
&= N^{(i)}(T) \delta_{\lfloor Ts \rfloor}^{(i)} \left(T^{-1} \sum_{t=1}^{T} \delta_{t/T}^{(i)\top} \delta_{t/T}^{(i)} \right)^{-1} \left(T^{-1} \sum_{t=1}^{T} \delta_{t/T}^{(i)\top} \left(\kappa^2(d) T^{2(d+1/2)} \right)^{-1/2} \tilde{z}_{\lfloor Ts \rfloor} \right) \\
&\Rightarrow \delta^{(i)}(s) \left(\int_0^1 \delta^{(i)}(s)^\top \delta^{(i)}(s) ds \right)^{-1} \left(\int_0^1 \delta^{(i)}(s)^\top B_{d+1}^{\underline{c}^{(i)}, \bar{c}^{(i)}}(s) ds \right)
\end{aligned}
$$

The above of course follows from Theorem 1 and the continuous mapping theorem. Thus, conclude that:

$$
\left(\kappa^2(d) T^{2(d+1/2)} \right)^{-1/2} \hat{\tilde{x}}_{\lfloor Ts \rfloor}^{(i)} \Rightarrow B_{d+1}^{\underline{c}^{(i)}, \bar{c}^{(i)}}(s) - \delta^{(i)}(s) \left(\int_0^1 \delta^{(i)}(r)^\top \delta^{(i)}(r) dr \right)^{-1} \left(\int_0^1 \delta^{(i)}(r)^\top B_{d+1}^{\underline{c}^{(i)}, \bar{c}^{(i)}}(r) dr \right)
$$

□

PROOF OF THEOREM 4. The proof combines the strategies found in the proofs of Theorem 1 and Theorem 3. First, the the proof of Theorem 1 is modified to show that under local alternatives, $\widetilde{X}_t^{(i)} \Rightarrow \chi_t^{(i)} + J_{d+1,c}^{c_t^{(i)},\bar{c}_t^{(i)}}(t)$. This is achieved if $\left(\kappa(d)T^{(d+1/2)}\right)^{-1} V_T^*(t)$ can be shown to converge to $J_{d+1,c}^{c_t^{(i)},\bar{c}_t^{(i)}}(t)$. To this end, following Phillips (1987b), note that in the context of near integrated alternatives,

$$V_T^*(t) = V_T(t) + \left(V_{[Tt]+1} - V_{[Tt]}\right) \frac{Tt - [Tt]}{\kappa(d)T^{(d+1/2)}}$$

where $V_T(t) = \sum_{k=1}^{[Tt]} e^{-c([Tt]-k)/T} \left(\kappa(d)T^{(d+1/2)}\right)^{-1} \triangle_+^{-d} v_k$. Nielsen (2009) demonstrated that $V_T(t)$ weakly converges to $J_{d+1,c}(t)$. What needs to be shown is that $V_T^*(t)$ converges to the same limit. Thus, observe the following.

$$\begin{aligned}
\sup_{t\in[0,1]} |V_T^*(t) - V_T(t)| &= \sup_{t\in[0,1]} \left| \left(V_{[Tt]+1} - V_{[Tt]}\right) \frac{Tt - [Tt]}{\kappa(d)T^{(d+1/2)}} \right| \\
&\leq \sup_{t\in[0,1]} \left| \sum_{k=1}^{[Tt]} \left(e^{-c([Tt]+1-k)/T} - e^{-c([Tt]-k)/T}\right) \left(\kappa(d)T^{(d+1/2)}\right)^{-1} \triangle_+^{-d} v_k \right| \\
&\leq \left(e^{-c/T} - 1\right) \sup_{t\in[0,1]} |V_T(t)| + \left(\kappa(d)T^{(d+1/2)}\right)^{-1} \max_{t=1,\ldots,T} \left|\triangle_+^{-d} v_t\right| \\
&= o_P(1)
\end{aligned}$$

The above follows because in the limit $V_T(t)$ is $O_P(1)$ and because the second term above converges by assumption (b). Thus conclude that $V_T^*(t)$ converges to $J_{d+1,c}^{c_t^{(i)},\bar{c}_t^{(i)}}(t)$. The convergence $\widetilde{X}_t^{(i)} \Rightarrow \chi_t^{(i)} + J_{d+1,c}^{c_t^{(i)},\bar{c}_t^{(i)}}(t)$ then follows by the same arguments used in the proof of Theorem 1 and Lemma 4. The theorem follows by adapting the proof of Theorem 3 to use the results obtained here so that the result follows by replacing in Theorem 3 any occurrence of $B_{d+1}^{c_t^{(i)},\bar{c}_t^{(i)}}(t)$ with $J_{d+1,c}^{c_t^{(i)},\bar{c}_t^{(i)}}(t)$. □